GEOMETRIC REPRESENTATION THEORY AND G-SIGNATURE

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Abstract. Let G be a finite group. To every smooth G-action on a compact, connected and oriented surface we can associate its data of singular orbits. The set of such data becomes an Abelian group $\mathbb{B}_G$ under the G-equivariant connected sum. We will show that the map which sends G to $\mathbb{B}_G$ is functorial and carries many features of the representation theory of finite groups. We will prove that $\mathbb{B}_G$ consists only of copies of $\mathbb{Z}$ and $\mathbb{Z}/2\mathbb{Z}$. Furthermore we will show that there is a surjection from the G-equivariant cobordism group of surface diffeomorphisms to $\mathbb{B}_G$.

We will define a G-signature θ which is related to the G-signature of Atiyah and Singer and prove that θ is injective on the copies of $\mathbb{Z}$ in $\mathbb{B}_G$.

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1. Introduction

Let G be a finite group and S_g a connected, oriented, compact, Riemann surface of genus g. In these notes we will study diffeomorphic G-actions on surfaces S_g up to isotopy, i.e., we will study all possible embeddings ϕ of the group G into the mapping class group Γ_g. The mapping class group is defined as follows. We write $\mathit{Diffeo}_+(S_g)$ for the group of orientation preserving diffeomorphisms of S_g with the C^∞-topology, and $\mathit{Diffeo}_0(S_g)$ for the connected component of the identity. Then we have $\Gamma_g = \mathit{Diffeo}_+(S_g)/\mathit{Diffeo}_0(S_g)$.

The mapping class group acts on the first homology group of S_g. This action preserves the intersection pairing and thus gives rise to a symplectic representation $\eta : \Gamma_g \to Sp_{2g}(\mathbb{R})$. The unitary group $U(g)$ is a maximal compact subgroup of $Sp_{2g}(\mathbb{R})$. Thus for any embedding $\phi : G \to \Gamma_g$ the map $\eta \circ \phi$ factors up to conjugation through $U(g) \subset$

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$Sp_{2g}(\mathbb{R})$ and this unitary representation is denoted by $\varphi(\phi)$. Hence φ assigns to every embedding ϕ a complex representation.

An important and interesting question is to determine the image of the map φ. The interest stems, among others, from the fact that one can use φ to deduce results about the cohomology of the mapping class group.

Using these ideas in the case where G is the cyclic group of prime order p, it is shown in [8] that the images of the symplectic classes $d_i \in H^{2i}(BSp(\mathbb{R}); \mathbb{Z})$ in $H^{2i}(\Gamma; \mathbb{Z})$ have infinite order. Here we consider the stable situation, i.e., Γ is the stable mapping class group.

Another result in [9] states that one can embed polynomial algebras in the cohomology of the stable mapping class group. For p a regular prime, we have $H^*(BSp(\mathbb{R}); \mathbb{F}_p) \cong \mathbb{F}_p[d_1, d_2, \dots]$, $deg\, d_i = 2i$ and the map $\eta^* : H^*(BSp(\mathbb{R}); \mathbb{F}_p) \to H^*(\Gamma; \mathbb{F}_p)$ is injective on the polynomial algebra $\mathbb{F}_p[(d_i)_{i \in J}]$, $J = \{i \in \mathbb{N} \mid i \equiv 1 \mod 2 \text{ or } i \equiv 0 \mod p-1\}$.

In this paper we want to study the image of the map φ. We will do this by turning φ into a group homomorphism. First we have to define a group structure on the set of diffeomorphic G-actions.

We will associate to every G-action its singular orbit data and add two such data by the G-equivariant connected sum (see section 2 for definitions). The resulting group of singular orbit data will be denoted by $\mathbb{B}_G$. In corollary 7 we will prove that the group $\mathbb{B}_G$ consists only of copies of $\mathbb{Z}$ and $\mathbb{Z}/2\mathbb{Z}$. We will see that the correspondence $G \mapsto \mathbb{B}_G$ is functorial and that it carries many features of the representation theory of finite groups. It has induction and restriction maps and also a double coset formula, therefore we could describe this functor as a geometric representation theory.

In section 4 we will define the map $\theta : \mathbb{B}_G \to R_{\mathbb{C}}G/D_G$, which is induced by φ. $R_{\mathbb{C}}G/D_G$ denotes a quotient of the complex representation ring (we will only consider the additive structure of the representation ring). We will prove results about the map θ and its image, recovering those obtained by Ewing [6] and Edmonds and Ewing [5] in the special case when G is a finite cyclic group. The main new result of this section is theorem 21, which states that θ is injective on the copies of $\mathbb{Z}$ in $\mathbb{B}_G$ for any finite group G.

In section 5 we will apply all the properties of the functor $\mathbb{B}$ and the map θ which have been introduced in the previous sections to the case $\mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z}$, p an odd prime. We will describe the image of θ by computing its index Δ in a subgroup $\mathbb{A}_G$ of the quotient $R_{\mathbb{C}}G/D_G$. This index will have the form $\Delta = (h_p^-)^{p+1} \cdot p^i$, where h_p^- is the first factor of the class number of $\mathbb{Q}(e^{2\pi i/p})$ and $1-p+k \leq i \leq k+(p-1)^2/2$ for some $k \in \mathbb{N}$.

In section 6 we will relate all the results we have obtained in the previous sections to G-equivariant cobordism. We will construct a surjection χ from the oriented G-equivariant cobordism group of surface diffeomorphisms Ω_G to $\mathbb{B}_G$. With this surjection θ becomes a G-signature, in the sense that $\theta \circ \chi$ is a homomorphism from Ω_G to a quotient of the representation ring of G.

It is section 6 which relates this paper to the papers of Ewing [6] and Edmonds and Ewing [5]. To prove his results, Ewing [6] essentially looks at the ring homomorphism from the $\mathbb{Z}/p\mathbb{Z}$-equivariant cobordism ring, p a prime, to the complex representation ring given by the G-signature of Atiyah and Singer [3] (AS-G-signature). In the second paper the authors use the AS-G-signature as a group homomorphism from the $\mathbb{Z}/n\mathbb{Z}$-equivariant cobordism group of surface diffeomorphisms to the complex representation ring. The value of the AS-G-signature for any finite group is computed by the Atiyah-Bott fixed point theorem [2] and it turns out that it is just $\varphi(\phi)-\overline{\varphi(\phi)}$; $\overline{\varphi(\phi)}$ denotes the complex conjugate representation.

We see that the G-signature given by θ and the AS-G-signature are closely related. In the special case of a finite cyclic group G, the results obtained in this paper are therefore similar to the ones obtained by Ewing and Edmonds in [6] and [5].

The AS-G-signature is very useful as long as the representations remain complex and non-real (i.e., almost all Abelian groups, see corollaries 22 and 24) but for real representations the AS-G-signature is zero. The approach described in this paper gives the possibility to construct a G-signature $\theta' : \mathbb{B}_G \to R_{\mathbb{C}}G/D'_G$ which is injective, even when $\varphi(\phi)$ is a real representation (see remark 5).

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2. The Group of Singular Orbit Data: Geometry

Let ϕ be an embedding of any finite group G into some mapping class group Γ_g. By Kerckhoff [11] any such embedding can be lifted to a homomorphism $G \to \mathit{Diffeo}_+(S_g)$, i.e., to an action of G on the surface S_g such that the elements of G act by orientation preserving diffeomorphisms. Let Gx denote the orbit of $x \in S_g$ under the action of G. The orbit is called singular if $|Gx| < |G|$, else regular. If the

orbit is singular, then there are elements of G which stabilize the point $x \in S_g$ and G_x denotes the stabilizer of G at x. It is proven by Accola [1, Lemma 4.10] that the stabilizers are cyclic subgroups of G. Let y be another element of the orbit Gx. Thus there is an element $a \in G$ such that $ax = y$ and the stabilizer of y is conjugate to G_x, i.e., $aG_xa^{-1} = G_y$. As the elements of G operate by orientation preserving diffeomorphisms there are only finitely many singular orbits. Let $x_i \in S_g$, $i = 1, ..., q$ be representatives of these orbits and ν_i the orders of the stabilizer groups G_{x_i}. Every group G_{x_i} has a generator $\gamma_i \in G$ such that γ_i acts by rotation through $2\pi/\nu_i$ on the tangent space at x_i. Similarly $a\gamma_i a^{-1}$, $a \in G$, generates $aG_{x_i}a^{-1}$ and acts *also* by rotation through $2\pi/\nu_i$ on the tangent space at ax_i. Thus in order to collect information about the singular orbits, it is enough to consider the conjugacy classes of the elements γ_i. Let $\hat{\gamma}_i$ denote the conjugacy class of γ_i in G. The extended singular orbit data (or extended data) of the embedding ϕ is then the collection

$$\sigma(\phi) = \left\{g; \hat{\gamma}_1, \dots, \hat{\gamma}_q\right\}_G$$

where g is the genus of the surface S_g and q is the number of singular orbits of the G-action; the conjugacy classes $\hat{\gamma}_i$ are unique up to order. This data depends a priori on the chosen lifting of ϕ, but we will show in the next lemma that the extended data is well defined for an embedding of G.

Lemma 1. *The extended data doesn't depend on the chosen lifting of the embedding and thus is well defined for any finite subgroup of the mapping class group.*

Proof. Symonds [13] proved that the extended data of a diffeomorphism of finite order depends only upon its isotopy class, and thus is well defined for an element of finite order of the mapping class group Γ_g. But if the extended data is given for the restriction to any element of a finite group, then it is also given for the whole group. □

The smaller collection $\left\{\hat{\gamma}_1, \dots, \hat{\gamma}_q\right\}_G$ will be called singular orbit data of the finite group G. Notice that there are infinitely many embeddings of G which give rise to the same singular orbit data $\left\{\hat{\gamma}_1, \dots, \hat{\gamma}_q\right\}_G$. In the sequel we will omit the subscript $_G$ if it is clear with respect to which group the conjugacy classes are taken.

We will now define an addition, the G-equivariant connected sum, on the set of singular orbit data W_G of a group G.

Let G act on a surface S_g with singular orbit data $\left\{\hat{\gamma}_1, \dots, \hat{\gamma}_q\right\}$ and on a surface S_h with singular orbit data $\left\{\hat{\beta}_1, \dots, \hat{\beta}_n\right\}$. Find discs D_1 in S_g and D_2 in S_h such that $\left\{aD_j\right\}_{a\in G}$ are mutually disjoint

for $j = 1, 2$. Then excise all discs $\left\{aD_j\right\}_{a \in G}$, $j = 1, 2$, from S_g and S_h and take a connected sum by matching $\partial(aD_1)$ to $\partial(aD_2)$ for all $a \in G$. The resulting surface $S_{g+h+|G|-1}$ has $|G|$ tubes joining S_g and S_h. The actions of G on S_g and S_h can be extended to an action on $S_{g+h+|G|-1}$ by permuting the tubes. The new action has a singular orbit data $\left\{\hat{\gamma}_1, \dots, \hat{\gamma}_q, \hat{\beta}_1, \dots, \hat{\beta}_n\right\}$. This construction on surfaces defines a monoid structure on the set W_G of singular orbit data with addition

$$\left\{\hat{\gamma}_1, \dots, \hat{\gamma}_q\right\} \oplus \left\{\hat{\beta}_1, \dots, \hat{\beta}_n\right\} := \left\{\hat{\gamma}_1, \dots, \hat{\gamma}_q, \hat{\beta}_1, \dots, \hat{\beta}_n\right\}$$

and zero element $\{\varnothing\}$, the free action. Next we will introduce a relation to turn this monoid into a group.

Suppose we have an action of G on a surface S_g with singular orbit data $\left\{\hat{\gamma}_1, \hat{\gamma}_1^{-1}, \hat{\gamma}_2, \dots, \hat{\gamma}_q\right\}$. Let $\nu = |\langle\gamma_1\rangle|$, then the conjugacy class $\hat{\gamma}_1$ gives rise to a singular orbit with representative x such that $a\gamma_1 a^{-1}$ acts by rotation through $2\pi/\nu$ on the tangent space at ax. On the other hand $\hat{\gamma}_1^{-1}$ gives rise to another singular orbit with representative z such that $a\gamma_1 a^{-1}$ acts by rotation through $-2\pi/\nu$ on the tangent space at az. Let T be a set of representatives for the $\langle\gamma_1\rangle$ left cosets of G. Find discs D_1 and D_2 around x and z respectively such that D_j is fixed by $\langle\gamma_1\rangle$, $j = 1, 2$, and $\cup_{j=1,2} \cup_{t \in T} \left\{tD_j\right\}$ are mutually disjoint. Then excise all discs $\left\{tD_j\right\}_{t \in T}$, $j = 1, 2$, from S_g and connect the boundaries $\partial(tD_1)$ with $\partial(tD_2)$ by means of tubes $S^1 \times [0, 1]$ for every $t \in T$. The resulting surface S_{g+w} has $w = |G|/\nu$ new handles. The action of G on S_g can be extended to S_{g+w} by permuting and rotating the new handles. This extended action yields the singular orbit data $\left\{\hat{\gamma}_2, \dots, \hat{\gamma}_q\right\}$. Pairs of singular orbits which have opposite rotation on the tangent spaces will be called cancelling pairs. The above process of eliminating cancelling pairs will be called reduction and if there are no such cancelling pairs left the singular orbit data is said to be in reduced form.

Now we can define the relation.

$$\left\{\hat{\gamma}_1, \dots, \hat{\gamma}_q\right\} \sim \left\{\hat{\beta}_1, \dots, \hat{\beta}_n\right\} :\Leftrightarrow \begin{cases} \text{The two singular orbit data} \\ \text{have the same reduced form} \end{cases}$$

This relation defines an equivalence relation on the set W_G of singular orbit data.

$$\mathbb{W}_G := W_G / \sim$$

The set $\mathbb{W}_G$ is not only a commutative monoid as W_G but contains also inverse elements and thus is a commutative group. The inverse element of $\left\{\hat{\gamma}_1, \dots, \hat{\gamma}_q\right\}$ is $\left\{\hat{\gamma}_1^{-1}, \dots, \hat{\gamma}_q^{-1}\right\}$ and the zero elements are cancelling pairs $\left\{\hat{\gamma}, \hat{\gamma}^{-1}\right\}$, $\gamma \in G$, and $\{\varnothing\}$ the free action.

3. The Group of Singular Orbit Data: Algebra

In the last section we defined a group structure on the singular orbit data. In this section we will define the same group but in an algebraic context. We will also find necessary and sufficient conditions for a q-tuple of conjugacy classes to define a singular orbit.

Every embedding $G \hookrightarrow \Gamma_g$ with extended data $\big\{g; \hat{\gamma}_1, \dots, \hat{\gamma}_q\big\}_G$ gives rise to a surjective homomorphism

$$\Big\langle a_1, b_1, \dots, a_h, b_h, \xi_1, \dots, \xi_q \,|\, \prod_{i=1}^{h} [a_i, b_i] \cdot \xi_1 \cdots \xi_q \Big\rangle \overset{p}{\twoheadrightarrow} G \tag{1}$$

such that the Riemann-Hurwitz equation

$$2g - 2 = |G|(2h - 2) + |G|\Big(\sum_{i=1}^{q} 1 - 1/\nu_i\Big) \tag{2}$$

is satisfied and $p(\xi_i) = \gamma_i$, $i = 1, \dots, q$. Here h denotes the genus of the quotient surface S_g/G, q the number of singular orbits and ν_i the order of the cyclic subgroup generated by $p(\xi_i)$.

By the discussion in Accola's book [1, Section 4.10 Branched Coverings] the converse is also true, for any surjective homomorphism p as in (1), there is an embedding $G \hookrightarrow \Gamma_g$ with extended data $\big\{g; \hat{\gamma}_1, \dots, \hat{\gamma}_q\big\}_G$ where g satisfies the Riemann-Hurwitz equation (2).

Furthermore any surjection p as in (1) gives rise to a q-tuple $\big(p(\xi_1), \dots, p(\xi_q)\big)$ such that $p(\xi_1) \cdots p(\xi_q) \in [G, G]$, the commutator subgroup of G.

Conversely for any q-tuple $\big(\gamma_1, \dots, \gamma_q\big) \in G \times \cdots \times G$, with $\gamma_1 \cdots \gamma_q \in [G, G]$ one can construct a surjection p as follows. Map ξ_i to γ_i for $i = 1, \dots, q$ and choose appropriate images for the a_i's and b_i's in G such that p becomes surjective and the relation $\prod_{i=1}^{h} [a_i, b_i] \cdot \xi_1 \cdots \xi_q$ is satisfied. This is always possible as one can choose h big enough. Observe that the construction of p doesn't depend on the ordering of the γ_i and neither on the representative of the conjugacy class of γ_i.

The above discussion shows that there is a correspondence between a singular orbit data $\big\{\hat{\gamma}_1, \dots, \hat{\gamma}_q\big\}_G$ and an unordered q-tuple of conjugacy classes $\big(\hat{\gamma}_1, \dots, \hat{\gamma}_q\big)$ with $\gamma_1 \cdots \gamma_q \in [G, G]$. Later we are going to make the correspondence more precise. But first consider the following map.

$$\begin{aligned} \Psi_q : G \times \cdots \times G \quad &\longrightarrow \quad \frac{G}{[G, G]} \\ \big(\gamma_1, \dots, \gamma_q\big) \quad &\longmapsto \quad \bar{\gamma}_1 \cdots \bar{\gamma}_q \end{aligned} \tag{3}$$

Ψ_q is a homomorphism and every element of $ker\,\Psi_q$ gives rise to a map p by the above considerations. It is clear by the definition of Ψ_q that $ker\,\Psi_q$ is invariant under permutation and conjugation of the components.

Definition 2.

$$\Lambda_G := \bigcup_{q=0}^{\infty} ker\,\Psi_q / \sim$$

Here $\sim$ means up to permutation and conjugation of the components and $ker\,\Psi_0$ is defined to be the empty tuple. An element of Λ_G will be denoted by $\left[\hat\gamma_1, \ldots, \hat\gamma_q\right]_G$, i.e., an unordered q-tuple of conjugacy classes of G, with $\gamma_1 \cdots \gamma_q \in [G, G]$. We will consider $ker\,\Psi_q/\sim$ as a subset of $ker\,\Psi_{q+1}/\sim$ by the inclusion $\left[\hat\gamma_1, \ldots, \hat\gamma_q\right]_G \mapsto \left[\hat\gamma_1, \ldots, \hat\gamma_q, 1\right]_G$, in particular all the q-tuples of ones $\left[1, \ldots, 1\right]_G$ will be the same as the empty tuple $[\varnothing]_G$.

One can now define a commutative monoid structure on Λ_G by

$$\left[\hat\gamma_1, \ldots, \hat\gamma_q\right]_G \oplus \left[\hat\beta_1, \ldots, \hat\beta_r\right]_G := \left[\hat\gamma_1, \ldots, \hat\gamma_q, \hat\beta_1, \ldots, \hat\beta_r\right]_G.$$

The associativity and commutativity of this addition is immediate, to obtain a group structure one has to introduce inverse elements. This is done by the next definition.

Definition 3.

$$\mathbb{B}_G := \Lambda_G / \left\langle [\hat\gamma, \hat\gamma^{-1}]_G \,,\, \gamma \in G \right\rangle$$

$\mathbb{B}_G$ is now a commutative group. The zero elements are $[\hat\gamma, \hat\gamma^{-1}]_G = [\varnothing]_G = 0$ and the inverse elements $\ominus\left[\hat\gamma_1, \ldots, \hat\gamma_q\right]_G = \left[\hat\gamma_1{}^{-1}, \ldots, \hat\gamma_q{}^{-1}\right]_G$. Now we can be more precise about the correspondence between the singular orbit data and the elements of $\mathbb{B}_G$.

Proposition 4. *There is an isomorphism of groups*

$$\begin{array}{ccc} \mathbb{B}_G & \xrightarrow{\sim} & \mathbb{W}_G \\ \left[\hat\gamma_1, \ldots, \hat\gamma_q\right]_G & \mapsto & \left\{\hat\gamma_1, \ldots, \hat\gamma_q\right\}_G \end{array}$$

given by the correspondence:
An element $\left[\hat\gamma_1, \ldots, \hat\gamma_q\right]_G$ *of* $\mathbb{B}_G$ *gives rise to a map* p *as in (1) which in turn defines an embedding* $G \hookrightarrow \Gamma_g$ *with singular orbit data* $\left\{\hat\gamma_1, \ldots, \hat\gamma_q\right\}_G$.

Proof. The map is well defined as the different choices of p give rise to the same singular orbit data.

By the construction it follows immediately that the map is a homomorphism and injective.

We can invert the correspondence by assigning to every singular orbit data $\{\hat{\gamma}_1, \dots, \hat{\gamma}_q\}_G$ a map p as in (1) and thus the element $[\hat{\gamma}_1, \dots, \hat{\gamma}_q]_G$ in $\mathbb{B}_G$. This shows that the map is surjective. It follows also that the correspondence is one-to-one. □

From now on we will refer to $\mathbb{B}_G$ as the group of singular orbit data and to elements $[\hat{\gamma}_1, \dots, \hat{\gamma}_q]_G$ as singular orbit data. If it is clear which group we mean, we will omit the subscript $_G$ and simply write $[\hat{\gamma}_1, \dots, \hat{\gamma}_q]$ for an element of $\mathbb{B}_G$.

For every group homomorphism $f : H \to G$ there is an induced homomorphism of their associated groups of singular orbit data.

$$\begin{aligned} \mathbb{B}_f : \mathbb{B}_H &\to \mathbb{B}_G \\ [\hat{\gamma}_1, \dots, \hat{\gamma}_q]_H &\mapsto [\widehat{f(\gamma_1)}, \dots, \widehat{f(\gamma_q)}]_G \end{aligned}$$

This map is well defined as $f(\gamma_1) \cdots f(\gamma_q) = f(\gamma_1 \cdots \gamma_q) \in f([H, H]) \subset [G, G]$. It is also true that $\mathbb{B}_{f \circ g} = \mathbb{B}_f \circ \mathbb{B}_g$ thus we have a covariant functor $\mathbb{B}$ from the category of finite groups into the category of Abelian groups.

$$\mathbb{B} : G \mapsto \mathbb{B}_G$$

Remark 1. If the map f is injective and the groups G and H are Abelian, then the map $\mathbb{B}_f$ is also injective.

Remark 2. If the map f is an isomorphism then $\mathbb{B}_f$ is also an isomorphism. In particular conjugation by an element of the group G will induce the identity map on $\mathbb{B}_G$, thus we have a map $Out(G) \to Aut(\mathbb{B}_G)$.

Remark 3. For every integer n we have the following action

$$n * [\hat{\gamma}_1, \dots, \hat{\gamma}_q] := [\hat{\gamma_1^n}, \dots, \hat{\gamma_q^n}].$$

This defines a map $\mathbb{Z} \xrightarrow{\lambda} End(\mathbb{B}_G)$ which preserves the multiplicative structure of $\mathbb{Z}$, $\lambda(nm) = \lambda(n) \circ \lambda(m)$, in particular $\lambda(0)$ is the zero map and $\lambda(1)$ is the identity map. If m is the order of G then the map λ factors through $\mathbb{Z}/m\mathbb{Z}$. The group of singular orbit data with its $\mathbb{Z}$-action is an invariant of the group G.

In the case where K is a subgroup of G then there is another map $\mathbb{B}res_K^G : \mathbb{B}_G \to \mathbb{B}_K$ called the restriction map. It is defined by restricting the singular orbits of G to the subgroup K. If the group G is Abelian we can give an explicit formula for this map, in the general case this formula becomes too complicated. We will omit the ˆ notation for Abelian groups.

$$\mathbb{B}res_K^G([\gamma_1, \dots, \gamma_q]) = [\gamma_{11}^{n_1}, \dots, \gamma_{1i_1}^{n_1}, \dots, \gamma_{q1}^{n_q}, \dots, \gamma_{qi_q}^{n_q}]$$

with $i_r = \frac{|G|\cdot|K\cap<\gamma_r>|}{|<\gamma_r>|\cdot|K|}$, $n_r = \min\{m|\gamma_r^m \in K\}$ and $\gamma_{rs} = \gamma_r$ for $s = 1, \dots, i_r$.

Let $i : H \to G$ be another subgroup, then there exists a double coset formula for singular orbit data.

Proposition 5. *Let H and K be subgroups of G and S a set of representatives for the (H, K) double cosets of G. For $s \in S$, let $H_s = sHs^{-1} \cap K$, which is a subgroup of K. The inclusions will be denoted by $i : H \to G$ and $j : H_s \to K$. Then we have the following equation*

$$\mathbb{B}res_K^G \circ \mathbb{B}_i = \bigoplus_{s\in S} \mathbb{B}_j \circ \mathbb{B}res_{H_s}^{sHs^{-1}} \circ \mathbb{B}_{f_s}$$

where f_s is just conjugation by s, i.e., $f_s(g) = sgs^{-1}$.

Proof. Let $\big[\hat{\gamma}, \dots\big]_H$ be a singular orbit data in $\mathbb{B}_H$, then we have $\mathbb{B}_i(\big[\hat{\gamma}, \dots\big]_H) = \big[\hat{\gamma}, \dots\big]_G$. First we consider the restriction on the level of singular orbits. Let Gx be the singular orbit corresponding to $\hat{\gamma}$. Thus γ operates by rotation through $2\pi/|\big\langle\gamma\big\rangle|$ on x. We can now partition the singular orbit Gx.

$$Gx = \bigcup_{s\in S} \bigcup_{g\in KsH} gx = \bigcup_{s\in S} \bigcup_{h\in H} Kshx = \bigcup_{s\in S} \bigcup_{h\in sHs^{-1}/H_s} Khsx$$

Thus we have

$$Gx = \bigcup_{s\in S} \bigcup_{h\in sHs^{-1}/H_s} Khsx$$

Note that the unions in the last equation are disjoint. Now we know how the restricted singular orbits look like. But what are the stabilizers of K at hsx?

$K \cap \big\langle hs\gamma s^{-1}h^{-1}\big\rangle = s^{-1}h^{-1}Khs \cap \big\langle\gamma\big\rangle = \big\langle\gamma_{hs}\big\rangle$ and γ_{hs} is unique such that it operates by rotation through $2\pi/|\big\langle\gamma_{hs}\big\rangle|$ on x. Thus we conclude:

$$\begin{aligned}
\mathbb{B}res_K^G \circ \mathbb{B}_i(\big[\hat{\gamma}, \dots\big]_H) &= \mathbb{B}res_K^G(\big[\hat{\gamma}, \dots\big]_G) \\
&= \big[\widehat{hs\gamma_{hs}(hs)}^{-1}, \dots\big]_K\ ,\ h \in sHs^{-1}/H_s\ ,\ s \in S
\end{aligned}$$

On the other hand we have

$$\begin{aligned}
\mathbb{B}_j \circ \mathbb{B}res_{H_s}^{sHs^{-1}} \circ \mathbb{B}_{f_s}\big[\hat{\gamma}, \dots\big]_H &= \mathbb{B}_j \circ \mathbb{B}res_{H_s}^{sHs^{-1}}\big[\widehat{s\gamma s^{-1}}, \dots\big]_{sHs^{-1}} = \\
\mathbb{B}_j\big[\widehat{hs\gamma_{hs}(hs)}^{-1}, \dots\big]_{H_s} &= \big[\widehat{hs\gamma_{hs}(hs)}^{-1}, \dots\big]_K\quad ,\ h \in sHs^{-1}/H_s
\end{aligned}$$

and both sides of the equation in the lemma are the same. $\square$

In this section we showed that many features of the representation theory of finite groups carry over to the functor $\mathbb{B}$. Thus we could say that $\mathbb{B}$ describes a geometric representation theory.

In the following examples the cyclic group of order m will be denoted by C_m.

Example 1. $G = C_2$; $\mathbb{B}_G \cong 0$.

Example 2. Let m be odd, $G = C_m = \langle x \rangle$ then $\mathbb{B}_G \cong \mathbb{Z}^{\frac{m-1}{2}}$ and a basis for $\mathbb{B}_G$ is given by $\left[x, x^i, x^{m-i-1}\right]$, $i = 1, \dots, \frac{m-1}{2}$.

Example 3. Let m be even, $G = C_m = \langle x \rangle$ then $\mathbb{B}_G \cong \mathbb{Z}^{\frac{m}{2}-1}$ and a basis for $\mathbb{B}_G$ is given by $\left[x, x^i, x^{m-i-1}\right]$, $i = 1, \dots, \frac{m}{2} - 1$.

Example 4. Let p be an odd prime, $G = C_p \times C_p = \langle x \rangle \times \langle y \rangle$ then $\mathbb{B}_G \cong \mathbb{Z}^{\frac{p^2-1}{2}}$ and a basis for $\mathbb{B}_G$ is given by $[x^j, y^i, x^{p-j}y^{p-i}]$, $j = 1, \dots, (p-1)/2$, $i = 1, \dots, p-1$; $[x, x^k, x^{p-k-1}]$, $k = 1, \dots, (p-1)/2$; $[y, y^l, y^{p-l-1}]$, $l = 1, \dots, (p-1)/2$.

For p the even prime, we have $\mathbb{B}_G \cong C_2$ and the generator is $\left[x, y, xy\right]$.

Example 5. Let $G = S_3 = C_3 \rtimes C_2$, $C_3 = \langle a \rangle$ and $C_2 = \langle b \rangle$. Then we have $[G, G] = C_3$, $\hat{a} = \{a, a^2\}$ and $\hat{b} = \{ab, a^2b, b\}$. The singular orbit data are then $\left[\hat{a}\right]$, $2 \cdot \left[\hat{a}\right] = \left[\hat{a}, \hat{a}\right] = \left[\hat{a}, \hat{a^2}\right] = 0$, $\left[\hat{b}, \hat{b}\right] = 0$. Thus the group of singular orbit data is generated by $\left[\hat{a}\right]$ and $\mathbb{B}_{S_3} = \left\langle \left[\hat{a}\right] \right\rangle \cong C_2$.

As $\mathbb{B}_{C_2} \cong 0$ the only maps which are of any interest are the maps $\mathbb{B}res^{S_3}_{C_3} : \mathbb{B}_{S_3} \to \mathbb{B}_{C_3}$ and $\mathbb{B}_i : \mathbb{B}_{C_3} \to \mathbb{B}_{S_3}$, where i is the inclusion $i : C_3 \hookrightarrow S_3$. The maps are given by $\mathbb{B}res^{S_3}_{C_3}([\hat{a}]_{S_3}) = [a, a^2]_{C_3} = 0$ and $\mathbb{B}_i([a, a, a]_{C_3}) = [\hat{a}, \hat{a}, \hat{a}]_{S_3} = [\hat{a}]_{S_3}$. Thus $\mathbb{B}res^{S_3}_{C_3}$ is just the zero map and $\mathbb{B}_i$ is the surjection of $\mathbb{Z}$ onto C_2.

Theorem 6. *The only torsion $\mathbb{B}_G$ contains, G any finite group, is two-torsion.*

Proof. Let $\alpha = \left[\hat{\gamma}_1, \dots, \hat{\gamma}_q\right]$ be an element of $\mathbb{B}_G$. Suppose α doesn't contain any cancelling pairs and it has finite order, i.e.,

$$n \cdot \left[\hat{\gamma}_1, \dots, \hat{\gamma}_q\right] = 0.$$

It follows that $n \cdot \alpha$ consists of cancelling pairs and thus $\hat{\gamma}_i = \hat{\gamma}_i^{-1}$ and n has to be even. But then we have already $2 \cdot \alpha = 0$, which proves the theorem. ∎

Corollary 7.

$$\mathbb{B}_G \cong \mathbb{Z} \times \cdots \times \mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \times \cdots \times \mathbb{Z}/2\mathbb{Z}$$

The copies of $\mathbb{Z}/2\mathbb{Z}$ are given by elements $\left[\hat{\gamma}_1, \dots, \hat{\gamma}_q\right]$ such that when in reduced form γ_i is conjugate to γ_i^{-1}, $i = 1, \dots, q$.

Proof. $\mathbb{B}_G$ is an Abelian group and by theorem 6 it contains only two-torsion. The group $\mathbb{B}_G$ is also finitely generated. Indeed, every element of $\mathbb{B}_G$ can be written as a linear combination of triples $\big[\hat{\gamma}_i, \hat{\gamma}_j, \hat{\gamma}_k\big]$ (this fact is proven in [10]) and there are only finitely many triples for a finite group G. The first assertion follows now by the structure theorem for Abelian groups.

If $\big[\hat{\gamma}_1, \dots, \hat{\gamma}_q\big]$ is a reduced element of order two then

$$\big[\hat{\gamma}_1, \dots, \hat{\gamma}_q\big] = \ominus\big[\hat{\gamma}_1, \dots, \hat{\gamma}_q\big] = \big[\hat{\gamma}_1{}^{-1}, \dots, \hat{\gamma}_q{}^{-1}\big]$$

and γ_i is conjugate to γ_i^{-1}, $i = 1, \dots, q$. The converse however is also true. □

Corollary 8. *Let G be an Abelian group which doesn't contain a copy of $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$, then there is no two torsion in $\mathbb{B}_G$.*

Proof. Assume that $\big[\gamma_1, \dots, \gamma_q\big] \in \mathbb{B}_G$ be a reduced element of order two. Thus $\gamma_j = \gamma_j^{-1}$, $j = 1, \dots, q$ and $q > 2$, i.e., there are at least two distinct involutions in G which commute. This contradicts the assumption on G. □

In [10] the author gives a complete computation of the group $\mathbb{B}_G$ for any finite group G .

4. Representations

In the previous sections we defined a group structure on the set of isotopy classes of diffeomorphic group actions. The resulting group was denoted by $\mathbb{B}_G$. In this section we will define a group homomorphism θ from the group $\mathbb{B}_G$ to a quotient of the complex representation ring of G. First we need a proposition which shows how to compute the representation $\varphi(\phi)$ from a given G-action ϕ. The map θ will then be induced by φ. We start with cyclic groups and follow the presentation in Farkas and Kra's book [7, pp 274].

4.1. **Cyclic Groups.** Let G be the cyclic group $C_n = \big\langle x \big\rangle$ and ϕ an embedding of G with extended data $\sigma(\phi) = \big\{g;\, x^{i_1}, \dots, x^{i_q}\big\}_{C_n}$. The representation $\varphi(\phi)$ will be a linear combination $\varphi(\phi) = \sum_{j=0}^{n-1} n_j \rho_i$ where $n_j \in \mathbb{N}$ and the ρ_i's, $i = 1, \dots, n-1$ are the one dimensional irreducible representations, such that ρ_1 is a faithful representation and $\rho_i = \rho_1^{\otimes i}$; ρ_0 is the one dimensional trivial representation. We will give a formula for the multiplicities n_j but first we need some notations: $gcd(n, i_s) = k_s$, $i_s = l_s k_s$ and thus l_s is prime to n. Let r_{sj} be the unique integer such that

$$1 \le r_{sj} \le n/k_s \quad \text{and} \quad r_{sj} \equiv j l_s \mod n/k_s\ , \ \ j = 1, \dots, n-1.$$

The formula in Farkas and Kra's book [7, pp 274] reads now:

$$\begin{aligned} n_j &= h-1+q-\big(\sum_{s=1}^{q} k_s r_{sj}\big)/n\ , \quad j=1,\dots,n-1 \qquad (4) \\ n_0 &= h. \end{aligned}$$

Here h denotes the genus of the quotient surface, it is well defined by the Riemann-Hurwitz equation (2). Thus we have proven the following lemma.

Lemma 9. *For a cyclic group C_n and an embedding ϕ the representation $\varphi(\phi)$ can be computed by its extended data $\sigma(\phi)=\big\{g;x^{i_1},\dots,x^{i_q}\big\}_{C_n}$. The formula for the multiplicities n_j is given by equation (4).*

In the sequel the regular representation of G will be denoted by ω_G or ω_m if the group is cyclic of order m.

Example 6. Let $G=C_n$. For a free action $\tilde{\phi}$ we have $\sigma(\tilde{\phi})=\big\{g;\varnothing\big\}_{C_n}$ and the Riemann-Hurwitz equation (2) reads $g=n(h-1)+1$. Equation (4) tells us that $\varphi(\tilde{\phi})=\rho_0+(h-1)\cdot\omega_n$.

Example 7. In this example we will consider the extended data $\big\{g;x^i, x^{n-i}\big\}_{C_n}$. In this situation we have $gcd(n,i)=k=gcd(n,n-i)$, $i=lk$, $n-i=(n/k-l)k$ thus $r_{1j}\equiv jl \mod n/k$ and $r_{2j}\equiv j(n/k-l) \mod n/k$. This implies $r_{2j}=n/k-r_{1j}$ if $r_{1j}\neq n/k$ and $r_{2j}=r_{1j}$ if $r_{1j}=n/k$. For the n_j we obtain

$$\begin{aligned} n_j &= h-1+2-(kr_{1j}/n)-(k(n/k-r_{1j})/n)=h\ ,\ j\not\equiv 0 \mod n/k \\ n_j &= h-1+2-1-1=h-1\ \ ,\ \ j\equiv 0 \mod n/k \\ n_0 &= h \end{aligned}$$

with Riemann-Hurwitz equation (2) $g=hn+(1-k)$. We can now express $\varphi(\phi)$ in terms of induced representations.

$$\varphi(\phi)=\sum_{j=0}^{n-1} n_j\rho_i = h\cdot Ind_{<0>}^{C_n}\rho_0+\rho_0-Ind_{<x^i>}^{C_n}\rho_0 = h\cdot\omega_n+\rho_0-Ind_{C_{n/k}}^{C_n}\rho_0$$

It follows readily by examples 6 and 7 that the additive subgroup of $R_{\mathbb{C}}C_n$ generated by all the representations induced by the extended data $\big\{g;x^i,x^{n-i}\big\}_{C_n}$, $i=0,\dots,n-1$, is the same as the subgroup

$$D_{C_n} := \big\langle Ind_H^{C_n}\rho_0 \ \big|\ H\leq C_n\big\rangle.$$

Lemma 10. *Every character with values in $\mathbb{Z}$ is a linear combination, with coefficients in $\mathbb{Z}$, of characters $Ind_H^{C_n}\rho_0$, where H runs through all the subgroups of C_n, i.e., $\langle Ind_H^{C_n}\rho_0 \;\big|\; H \le C_n \rangle = R_{\mathbb{Q}}C_n$.*

Proof. For each divisor d of n, let

$$\begin{aligned} \chi_d &= \sum_{l=1, gcd(l,d)=1}^{d-1} \rho_l^{\otimes n/d} \;,\;\; d \neq 1 \\ \chi_1 &= \rho_0 \;,\;\; d = 1 \end{aligned}$$

Then the χ_d form an orthogonal basis of $R_{\mathbb{Q}}C_n$. On the other hand we have

$$Ind_{C_{n/d}}^{C_n}\rho_0 = \sum_{l=1}^{d} \rho_l^{\otimes n/d} = \sum_{d'|d} \sum_{l=1, gcd(l,d')=1}^{d'-1} \rho_l^{\otimes n/d'} = \sum_{d'|d} \chi_{d'}.$$

By induction we can now prove that the χ_d are linear combinations of the $Ind_{C_{n/d'}}^{C_n}\rho_0$, $d'|d$, with integer coefficients. Thus the $Ind_{C_{n/d}}^{C_n}\rho_0$, $d|n$, form a basis of $R_{\mathbb{Q}}C_n$. (See also exercise 13.1 of Serre's book [12].) ☐

Let $\sigma(\tilde{\phi})$ and $\sigma(\bar{\phi})$ be two extended data which have the same singular orbits. Then by equation (4) their associated unitary representations $\varphi(\tilde{\phi})$ and $\varphi(\bar{\phi})$ differ only by a multiple of the regular representation ω_G. Thus by example 6 there is a well defined map

$$\Lambda_{C_n} \to R_{\mathbb{C}}C_n / \langle \rho_0, \omega_n \rangle$$

which is induced by φ. This map is even a homomorphism of Abelian monoids. Indeed, given two extended data $\sigma(\phi) = \{g; x^{i_1}, \dots, x^{i_q}\}_{C_n}$ and $\sigma(\phi') = \{g'; x^{j_1}, \dots, x^{j_w}\}_{C_n}$. Then again by equation (4) we can see that

$$\varphi(\phi) + \varphi(\phi') \equiv \varphi(\tau) \quad \text{up to multiples of } \rho_0 \text{ and } \omega_n,$$

where τ gives rise to an extended data $\{\bar{g}; x^{i_1}, \dots, x^{i_q}, x^{j_1}, \dots, x^{j_w}\}_{C_n}$.

Example 7 shows that the zero-elements of $\mathbb{B}_{C_n}$ are mapped into D_{C_n} thus we obtain a well defined homomorphism of Abelian groups which is induced by φ:

$$\theta : \mathbb{B}_{C_n} \to R_{\mathbb{C}}C_n / D_{C_n} =: \widetilde{R_{\mathbb{C}}C_n}.$$

Let H be a subgroup of C_n. Then the homomorphisms $Res_H^{C_n} : R_{\mathbb{C}}C_n \to R_{\mathbb{C}}H$ and $Ind_H^{C_n} : R_{\mathbb{C}}H \to R_{\mathbb{C}}C_n$ map D_{C_n} into D_H and D_H into D_{C_n} respectively. Thus they induce well defined homomorphisms $\widetilde{Res}_H^{C_n} : \widetilde{R_{\mathbb{C}}C_n} \to \widetilde{R_{\mathbb{C}}H}$ and $\widetilde{Ind}_H^{C_n} : \widetilde{R_{\mathbb{C}}H} \to \widetilde{R_{\mathbb{C}}C_n}$.

Proposition 11. *For a cyclic group G and a subgroup $H < G$ the following diagram commutes.*

$$\begin{array}{ccc} \mathbb{B}_H & \overset{\mathbb{B}res_H^G}{\longleftarrow} & \mathbb{B}_G \\ \downarrow & & \downarrow \\ \widetilde{R_{\mathbb{C}}H} & \overset{\widetilde{Res}_H^G}{\longleftarrow} & \widetilde{R_{\mathbb{C}}G} \end{array}$$

Proof. Let $\alpha \in \mathbb{B}_G$ and $\phi_\alpha : G \hookrightarrow \Gamma_g$ denote a corresponding embedding. We can now easily describe a possible embedding for $\mathbb{B}res_H^G\alpha$, namely $\phi_{\mathbb{B}res_H^G\alpha} = \phi_\alpha|_H : H < G \hookrightarrow \Gamma_g$. From this we deduce $Res_H^G\varphi(\phi_\alpha) = \varphi(\phi_{\mathbb{B}res_H^G\alpha})$ which proves the proposition. $\square$

Proposition 12. *For a cyclic group G and a subgroup $H < G$ the following diagram commutes.*

$$\begin{array}{ccc} \mathbb{B}_H & \overset{\mathbb{B}_i}{\longrightarrow} & \mathbb{B}_G \\ \downarrow & & \downarrow \\ \widetilde{R_{\mathbb{C}}H} & \overset{\widetilde{Ind}_H^G}{\longrightarrow} & \widetilde{R_{\mathbb{C}}G} \end{array}$$

Proof. For the proof we need some explicit computations, so we choose $G = C_n = \langle x \rangle$ and $H = C_l = \langle y \rangle$ where $l|n$ with the inclusion $i : H \to G$, $y \mapsto x^{n/l}$. In the sequel we will use the notation r_{sj}^l, n_j^l and r_{sj}^n, n_j^n to denote integers dealing with the groups C_l and C_n respectively.

Let $\alpha = [y^{i_1}, \dots, y^{i_q}]$ be an element of $\mathbb{B}_H$. Then with the notation of equation (4) we obtain: $gcd(l, i_s) = k_s$, $s = 1, \dots, q$, $i_s = k_s u_s$ and thus

$$r_{sj}^l \equiv j u_s \mod l/k_s \ , \ j = 1, \dots, l-1$$

where $1 \leq r_{sj}^l \leq l/k_s$. By equation (4) we obtain for the multiplicities of the representations:

$$-n_j^l \equiv \Big(\sum_{s=1}^{q} k_s r_{sj}^l\Big)/l \ , \text{ up to multiples of } \omega_l$$

where $j = 1, \dots, l-1$.

On the other hand we have $\mathbb{B}_i([y^{i_1}, \dots, y^{i_q}]) = [x^{i_1 n/l}, \dots, x^{i_q n/l}]$ which leads to: $gcd(n, i_s n/l) = k_s n/l$, $s = 1, \dots, q$, $i_s n/l = k_s u_s n/l$ and thus

$$r_{sj}^n \equiv j u_s \mod n/(k_s n/l) = l/k_s \ , \ j = 1, \dots, n-1$$

where $1 \leq r^n_{sj} \leq n/(k_s n/l) = l/k_s$. By equation (4) we obtain for the multiplicities of the representations:

$$\begin{aligned} -n^n_j &\equiv \big(\sum_{s=1}^{q} k_s r^n_{sj} n/l\big)/n \text{ , up to multiples of } \omega_n \\ &= \big(\sum_{s=1}^{q} k_s r^l_{sj}\big)/l \end{aligned}$$

where $j = 1, \ldots, n-1$.

This yields $n^l_j \equiv n^n_i$, up to multiples of the regular representation, for all $j \equiv i \mod l$, $j = 1, \ldots, l-1$, $i = 1, \ldots, n-1$. But this is exactly the property of an induced representation, i.e., $\varphi(\phi_{\mathbb{B}_i \alpha}) \equiv Ind^G_H \varphi(\phi_\alpha)$, up to multiples of ω_G and ρ_0. Thus we have proven the assertion $\theta \circ \mathbb{B}_i([y^{i_1}, \ldots, y^{i_q}]) = \widetilde{Ind}^G_H \circ \theta([y^{i_1}, \ldots, y^{i_q}])$. □

By the notation $\bar{a}$ for an element $a \in \widetilde{R_{\mathbb{C}} G}$ we mean the following. Let ξ be a representative of the class a in $R_{\mathbb{C}} G$ then the complex conjugate $\bar{\xi}$ is a representative of $\bar{a}$.

Proposition 13. *Let $\alpha \in \mathbb{B}_{C_n}$. Then the following holds.*

$$\theta(-\alpha) = \overline{\theta(\alpha)}$$

Proof. In the following we will use the notation of equation (4). Let $\alpha = [x^{i_1}, \ldots, x^{i_q}]$, then $-\alpha = [x^{-i_1}, \ldots, x^{-i_q}]$ thus $r^\alpha_{sj} \equiv j l_s \mod n/k_s$ and $r^{-\alpha}_{sj} \equiv j(-l_s) \equiv (n-j) l_s \mod n/k_s$. Insert this in equation (4) and we obtain

$$n^\alpha_j = n^{-\alpha}_{n-j}$$

which yields the proof. □

From proposition 13 we deduce that $-\theta(\alpha) = \overline{\theta(\alpha)}$ and thus θ maps into

$$\mathbb{A}_G := \big\{a \in \widetilde{R_{\mathbb{C}} G} | a + \bar{a} = 0\big\}.$$

Remark 4. Proposition 13 follows also from the Lefschetz Fixed-Point-Formula which states that the trace of $\varphi(\phi) + \overline{\varphi(\phi)}$ equals two minus the number of fixed points.

Theorem 14. *The map $\theta : \mathbb{B}_{C_n} \to \widetilde{R_{\mathbb{C}} C_n}$ is injective and the index of the image of θ in $\mathbb{A}_{C_n}$ is finite.*

Proof. First we will prove the injectivity of θ. Let ϕ be an embedding of C_n and α the corresponding element of ϕ in $\mathbb{B}_{C_n}$.
Assume that $\theta(\alpha) = 0$. By lemma 10 the representation $\varphi(\phi)$ is integral and as the C_n-signature of Atiyah and Singer [3, p. 578] is given by

$sgn(\phi) = \varphi(\phi) - \overline{\varphi(\phi)}$ we have $sgn(\phi) = 0$. By a theorem of Edmonds and Ewing [5, Theorem 3.2] this implies that the fixed points of α are cancelling. (Note that this theorem doesn't say anything about the other singular orbits.) We can now reduce α to a fixed point free element of $\mathbb{B}_{C_n}$.

Let H be a maximal proper subgroup of C_n. By proposition 11 we have $\theta(\mathbb{B}res_H^G \alpha) = \widetilde{Res}_H^G \theta(\alpha) = 0$. The same argument as above tells us that the fixed points of $\mathbb{B}res_H^G \alpha$ cancel, i.e., the singular orbits of α which have H as stabilizer cancel. We again reduce α and iterate this procedure by restricting to smaller and smaller subgroups of C_n. Finally we end up with a free action and $\alpha = 0$.

It is not difficult to see that $\mathbb{A}_{C_n}$ has rank $(n-1)/2$ for n odd and $n/2 - 1$ for n even. By examples 2 and 3, $\mathbb{B}_{C_n}$ has the same rank and as θ is injective the index of $\theta(\mathbb{B}_{C_n})$ in $\mathbb{A}_{C_n}$ is finite. $\square$

Proposition 15. *For C_p the cyclic group of prime order p the index of $\theta(\mathbb{B}_{C_p})$ in $\mathbb{A}_{C_p}$ is h_p^-, the first factor of the class number of $\mathbb{Q}(e^{2\pi i/p})$.*

Proof. The original proof can be found in Ewing's paper [6, Theorem 3.2]. A proof which uses singular orbits and lies in the spirit of this paper is given in [8, Proposition 4] $\square$

In view of proposition 15 it is natural to ask if there is a relation between the finite index of theorem 14 and the class number of $\mathbb{Q}(e^{2\pi i/n})$. This question is not settled for the moment but one can say the following. For m an odd integer there exists a subgroup M of $\mathbb{B}_{C_m}$ and a subgroup N of $\mathbb{A}_{C_m}$ such that the index of M in N is the class number of $\mathbb{Q}(e^{2\pi i/m})$.

4.2. **Finite Groups.** In this section we will prove similar properties for arbitrary finite groups as we did for cyclic groups in section 4.1.

Proposition 16. *Let $\phi : G \hookrightarrow \Gamma_g$ be an embedding. The corresponding representation $\varphi(\phi) : G \to U(g)$ can be computed by $\sigma(\phi) = \{g; \hat{\gamma}_1, \ldots, \hat{\gamma}_q\}_G$.*

Proof. Let $\{G_i\}_{i \in I}$ be the family of the maximal cyclic subgroups of G. If a representation of G is known when restricted to all the maximal cyclic subgroups then the representation itself is known.

We can restrict the extended data to any G_i and we obtain a data $Res_{G_i}^G \sigma(\phi) = \{g; \hat{\beta}_1, \ldots, \hat{\beta}_n\}_{G_i}$ which belongs to the restricted embedding $\phi_i = \phi|_{G_i} : G_i \hookrightarrow G \hookrightarrow \Gamma_g$.

By lemma 9 we can compute $\varphi(\phi_i)$ for every G_i and thus the representation $\varphi(\phi)$ is given by the extended data. $\square$

In the following ρ_0^G will always denote the one dimensional trivial representation of G, usually we will just write ρ_0 when there is no confusion possible.

Example 8. Let G be a finite group and $\sigma(\phi) = \big\{g; \varnothing\big\}_G$ an extended data for a free action with Riemann-Hurwitz equation (2) $g = 1 + |G|(h-1)$. Restricting this extended data to any cyclic subgroup H gives $Res_H^G(\sigma(\phi)) = \big\{g; \varnothing\big\}_H$ and Riemann-Hurwitz equation $g = 1 + |H|(\tilde{h}-1)$. Thus we have by example 6 that $\varphi(\phi|_H) = \rho_0 + \omega_H(h-1)|G|/|H|$. We can now reconstruct the representation $\varphi(\phi)$ and obtain

$$\varphi(\phi) = \rho_0 + (h-1)\cdot\omega_G.$$

Proposition 17. *The representation $\varphi(\phi)$ associated to $\sigma(\phi) = \big\{g; \hat{\gamma}, \hat{\gamma}^{-1}\big\}_G$ for any finite group G and any element $\gamma \in G$ equals*

$$\xi = h\cdot\omega_G + \rho_0 - Ind_{<\gamma>}^G \rho_0.$$

Proof. Let ν denote the order of the element γ. Then we can write the Riemann-Hurwitz equation in the form $g = |G|h + 1 - |G|/\nu$. For H a cyclic subgroup of G let $t_i \in G$ $i = 1, \dots, |G|/|H|$ be the left coset representatives; then we can write $\beta_i = t_i\gamma^{r_i}t_i^{-1}$, $r_i = \min\{s \in \mathbb{N} | t_i\gamma^s t_i^{-1} \in H\}$. We can renumber the β_i's such that $\beta_i \neq 1$ for $i = 1, \dots, t$ and $\beta_i = 1$ for $i = t+1, \dots, |G|/|H|$. Restricting ϕ to H gives rise to the following extended data and representation (see example 7).

$$\begin{aligned}
Res_H^G\sigma(\phi) &= \big\{g; \hat{\beta}_1, \hat{\beta}_1^{-1}, \dots, \hat{\beta}_t, \hat{\beta}_t^{-1}\big\}_H \\
\varphi(\phi|_H) &= \big((h-1)|G|/|H| + t\big)\cdot\omega_H + \rho_0 - \sum_{i=1}^{t} Ind_{<\beta_i>}^H \rho_0 \\
&= |G|/|H|\cdot h\cdot\omega_H + \rho_0 - \sum_{i=1}^{|G|/|H|} Ind_{<\beta_i>}^H \rho_0 \\
&= Res_H^G \xi
\end{aligned}$$

As $\varphi(\phi)$ and ξ coincide on the cyclic subgroups they are the same representation. □

We can define the following subgroups of $R_{\mathbb{C}}G$

$$\begin{aligned}
D_G &:= \big\langle Ind_H^G\rho_0 \,|\, H = G \text{ and } H < G\ ,\ H \text{ cyclic}\big\rangle \\
E_G &:= \big\langle Ind_H^G\rho_0 \,|\, H \leq G\big\rangle.
\end{aligned}$$

By example 8 and proposition 17, D_G is generated by representations $\varphi(\phi)$ where ϕ runs through free actions and actions which consist of

cancelling pairs. Notice that $D_G = E_G$ if G is cyclic. But in general D_G is strictly contained in E_G. In addition lemma 10 is not true for arbitrary groups; thus we have

$$D_G < E_G < R_{\mathbb{Q}}G.$$

In the same way as for cyclic groups, example 8 shows that there is a well defined map

$$\Lambda_G \to R_{\mathbb{C}}G/\langle \rho_0, \omega_G \rangle$$

induced by φ and by proposition 17 this map extends to

$$\tilde{\theta} : \mathbb{B}_G \to R_{\mathbb{C}}G/D_G.$$

In order to have commutative squares as in propositions 11 and 12 we need another map

$$\theta : \mathbb{B}_G \to R_{\mathbb{C}}G/E_G =: \widetilde{R_{\mathbb{C}}G}.$$

The map θ factors through $\tilde{\theta}$ and for G a cyclic group they are the same as we have in this case $D_G = E_G$. For a subgroup H of G the restriction map $\widetilde{Res}_H^G : \widetilde{R_{\mathbb{C}}G} \to \widetilde{R_{\mathbb{C}}H}$ and induction map $\widetilde{Ind}_H^G : \widetilde{R_{\mathbb{C}}H} \to \widetilde{R_{\mathbb{C}}G}$ are again well defined as $Res_H^G E_G \subset E_H$ and $Ind_H^G E_H \subset E_G$.

Proposition 18. *For a finite group G and a subgroup $H < G$ the following diagram commutes.*

$$\begin{array}{ccc} \mathbb{B}_H & \overset{\mathbb{B}res_H^G}{\longleftarrow} & \mathbb{B}_G \\ \downarrow & & \downarrow \\ \widetilde{R_{\mathbb{C}}H} & \overset{\widetilde{Res}_H^G}{\longleftarrow} & \widetilde{R_{\mathbb{C}}G} \end{array}$$

Proof. The proof is analogous to the one for proposition 11. □

Lemma 19. *Let G be any finite group with subgroup H and $f_s : H \to sHs^{-1}$ conjugation by $s \in G$. Then there are induced maps $\mathbb{B}_{f_s} : \mathbb{B}_H \to \mathbb{B}_{sHs^{-1}}$, $f_{s*} : R_{\mathbb{C}}H \to R_{\mathbb{C}}sHs^{-1}$ and $\tilde{f}_{s*} : \widetilde{R_{\mathbb{C}}H} \to \widetilde{R_{\mathbb{C}}sHs^{-1}}$. The map f_{s*} is given by $f_{s*}(\rho)(a) = \rho(s^{-1}as)$, $\rho \in R_{\mathbb{C}}H$, $a \in sHs^{-1}$, and the map $\tilde{f}_{s*}$ is induced by f_{s*} on the quotient $\widetilde{R_{\mathbb{C}}H}$. These maps satisfy the following equations:*

$$\begin{aligned} \varphi(\phi_{\mathbb{B}_{f_s}\alpha}) &\equiv f_{s*}(\varphi(\phi_\alpha)) \text{ , up to multiples of } \omega_{sHs^{-1}} \\ \theta \circ \mathbb{B}_{f_s}(\alpha) &= \tilde{f}_{s*} \circ \theta(\alpha) \end{aligned}$$

for $\alpha \in \mathbb{B}_H$, $\mathbb{B}_{f_s}(\alpha) \in \mathbb{B}_{sHs^{-1}}$ with corresponding embeddings ϕ_α and $\phi_{\mathbb{B}_{f_s}\alpha}$ respectively.

Proof. The second equation follows directly from the first one. To prove the first one just note that:

$$\varphi(\phi_{\mathbb{B}_{f_s}\alpha}) \equiv \varphi(f_{s*}(\phi_\alpha)) = f_{s*}(\varphi(\phi_\alpha)) \text{ , up to multiples of } \omega_{sHs^{-1}},$$

where of course $f_{s*}(\phi_\alpha)(a) = \phi_\alpha(s^{-1}as)$. ☐

Proposition 20. *For a finite group G and a subgroup $H < G$ the following diagram commutes.*

$$\begin{array}{ccc} \mathbb{B}_H & \stackrel{\mathbb{B}_i}{\rightarrow} & \mathbb{B}_G \\ \downarrow & & \downarrow \\ \widetilde{R_{\mathbb{C}}H} & \stackrel{\widetilde{Ind}_H^G}{\rightarrow} & \widetilde{R_{\mathbb{C}}G} \end{array}$$

Proof. To prove the statement of the proposition it is enough to prove the following equation:

$$\varphi(\phi_{\mathbb{B}_i\alpha}) \equiv Ind_H^G\varphi(\phi_\alpha) \text{ , up to multiples of } \omega_G \text{ and } \rho_0^G \tag{5}$$

for some elements $\mathbb{B}_i\alpha \in \mathbb{B}_G$, $\alpha \in \mathbb{B}_H$ with corresponding embeddings $\phi_{\mathbb{B}_i\alpha}$ and ϕ_α respectively. To do this, we will restrict both sides of equivalence (5) to an arbitrary cyclic subgroup K of G and then show that they coincide, up to multiples of ω_K and ρ_0^K. This will then imply that the original equivalence (5) is correct.

Let K be a cyclic subgroup of G and S a set of representatives for the (H, K) double cosets of G. For $s \in S$, let $H_s = sHs^{-1} \cap K$, which is a subgroup of K. The inclusions will be denoted by $i : H \to G$ and $j : H_s \to K$. f_s will be conjugation by s, i.e., $f_s(g) = sgs^{-1}$.

In the following we will write $\varphi \circ F(\alpha)$ instead of $\varphi(\phi_{F(\alpha)})$ for any map F. All the equivalences will be up to multiples of ω_K and ρ_0^K.

First we will apply Res_K^G to the right hand side of equivalence (5). Using proposition 18, lemma 19 and the well known double coset formula for representations we obtain

$$\begin{aligned} Res_K^G \circ Ind_H^G \circ \varphi &= \bigoplus_{s\in S} Ind_{H_s}^K \circ Res_{H_s}^{sHs^{-1}} \circ f_{s*} \circ \varphi \\ &\equiv \bigoplus_{s\in S} Ind_{H_s}^K \circ \varphi \circ \mathbb{B}res_{H_s}^{sHs^{-1}} \circ \mathbb{B}_{f_s}. \end{aligned}$$

Now we apply Res_K^G to the left hand side of equivalence (5). Using propositions 18 and 5 we obtain

$$\begin{aligned} Res_K^G \circ \varphi \circ \mathbb{B}_i &\equiv \varphi \circ \mathbb{B}res_K^G \circ \mathbb{B}_i \equiv \varphi \circ \bigoplus_{s\in S} \mathbb{B}_j \circ \mathbb{B}res_{H_s}^{sHs^{-1}} \circ \mathbb{B}_{f_s} \\ &\equiv \bigoplus_{s\in S} \varphi \circ \mathbb{B}_j \circ \mathbb{B}res_{H_s}^{sHs^{-1}} \circ \mathbb{B}_{f_s}. \end{aligned}$$

By the proof of proposition 12 we know that $\varphi \circ \mathbb{B}_j \equiv Ind_{H_s}^K \circ \varphi$, up to multiples of ω_K, for cyclic groups K. Thus both sides of equivalence (5) coincide up to multiples of ω_G and ρ_0^G and the proposition is true. $\square$

Recall from corollary 7 that $\mathbb{B}_G$ consists only of copies of $\mathbb{Z}$ and $\mathbb{Z}/2\mathbb{Z}$.

Theorem 21. *The map θ is injective on the copies of $\mathbb{Z}$ in $\mathbb{B}_G$.*

Proof. Let $\left[\hat{\gamma_1}, \dots, \hat{\gamma_q}\right]$ be a reduced element of infinite order in $\mathbb{B}_G$, i.e., γ_i is not conjugate to γ_j^{-1}, $i \neq j$, $i, j = 1, \dots, q$ and there exists at least one γ_{i_0}, $1 \leq i_0 \leq q$, which is not conjugate to its inverse. There is also an $s = 1, \dots, q$ such that we can renumber the γ_i's in the following way: $i_0 = 1$, $\langle \gamma_i \rangle$ is conjugate to $\langle \gamma_1 \rangle$, $i \leq s \leq q$, and $\langle \gamma_j \rangle$ is not conjugate to $\langle \gamma_1 \rangle$, $s < j \leq q$. Then we have that γ_i is not conjugate to γ_j^{-1}, $1 \leq i, j \leq s$.

Notice that γ_1 can be chosen such that $\langle \gamma_1 \rangle$ is maximal with respect to the other cyclic subgroups $\langle \gamma_i \rangle$, $i = s+1, \dots, q$, up to conjugation.

The singular orbit data $\alpha = \mathbb{B}res^G_{<\gamma_1>}\left[\hat{\gamma_1}, \dots, \hat{\gamma_q}\right]$ has then fixed points which don't cancel and thus $\alpha \neq 0$. (Notice that the other singular orbits of α could cancel.) By theorem 14 we know that $\theta(\alpha) \neq 0$ and by proposition 18 we deduce

$$\widetilde{Res}^G_{<\gamma_1>}\theta(\left[\hat{\gamma_1}, \dots, \hat{\gamma_q}\right]) = \theta(\alpha) \neq 0$$

and thus $\theta(\left[\hat{\gamma_1}, \dots, \hat{\gamma_q}\right]) \neq 0$ $\square$

Corollary 22. *Let G be an Abelian group which doesn't contain a copy of $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. The map θ is then injective.*

Proof. By corollary 8, $\mathbb{B}_G$ is torsion free and by theorem 21, θ is injective on the copies of $\mathbb{Z}$. $\square$

Proposition 23. *For G a finite group, $\theta : \mathbb{B}_G \to \widetilde{R_{\mathbb{C}}G}$ maps into*

$$\mathbb{A}_G := \left\{a \in \widetilde{R_{\mathbb{C}}G} | a + \bar{a} = 0\right\},$$

i.e., $\theta(-\alpha) = \overline{\theta(\alpha)}$.

Proof. Let α be an element of $\mathbb{B}_G$ with corresponding embedding ϕ_α and unitary representation $\varphi(\phi_\alpha)$. Changing the orientation of the surface is equivalent with taking the transposed symplectic structure in the first homology group of the surface. On the other hand transposition in $Sp_{2n}(\mathbb{R})$ induces complex conjugation in its unitary subgroup $U(n) \subset Sp_{2n}(\mathbb{R})$.

Thus to the element $-\alpha$ corresponds the unitary representation $\overline{\varphi(\phi_\alpha)}$ and we obtain $-\theta(\alpha) = \theta(-\alpha) = \overline{\theta(\alpha)}$. $\square$

Corollary 24. *For an element α of order two in $\mathbb{B}_G$ we have*

$$\theta(\alpha) = \overline{\theta(\alpha)}$$

and for an element β of infinite order

$$\theta(\beta) \neq \overline{\theta(\beta)}.$$

Proof. As $\alpha = -\alpha$ we have by proposition 23, $\theta(\alpha) = \theta(-\alpha) = \overline{\theta(\alpha)}$.

In the second case it follows that $\beta \neq -\beta$ and thus by theorem 21 and proposition 23, $\theta(\beta) \neq \theta(-\beta) = \overline{\theta(\beta)}$. $\square$

Remark 5. For $G = S_3$, the symmetric group on three letters, we have $D_G = R_{\mathbb{Q}}G = R_{\mathbb{C}}G$. Thus θ is the zero map.

Note that the G-signature defined by Atiyah and Singer $sgn(\phi) = \varphi(\phi) - \overline{\varphi(\phi)}$ is also zero. But with the approach we chose, it is now possible to modify θ such that it becomes injective again.

For S_3 let χ_0 be the trivial representation, χ_1 the one dimensional non-trivial representation and χ_2 the two dimensional irreducible representation. Recall the notation from example 5 and let

$$D'_{S_3} = \left\langle Ind_{<a>}^{S_3}\rho_0\,,\, Ind_{<1>}^{S_3}\rho_0\,,\, Ind_{S_3}^{S_3}\rho_0 \right\rangle = \left\langle \chi_0+\chi_1\,,\, \chi_0+\chi_1+2\chi_2\,,\, \chi_0 \right\rangle.$$

Then we have that the class of χ_2 generates $R_{\mathbb{C}}S_3/D'_{S_3} \cong C_2$. We know that $\mathbb{B}_{S_3} \cong \left\langle \left[\hat{a}\right] \right\rangle \cong C_2$; thus it remains to show that $\theta' : \mathbb{B}_{S_3} \to R_{\mathbb{C}}S_3/D'_{S_3}$ is non zero on $\left[\hat{a}\right]$.

Let $\sigma(\phi) = \left\{3; \hat{a}\right\}_{S_3}$ then we find that $\varphi(\phi) = \chi_0 + \chi_2$ and thus θ' maps $\left[\hat{a}\right]$ to the class of χ_2 and θ' is an isomorphism.

Note that D'_{S_3} consists of the minimal relations in order to make θ' a group homomorphism. It consists of the representations coming from the free actions and the cancelling pair $\left[\hat{a}, \hat{a}\right]_{S_3}$.

In this situation the maps $\mathbb{B}_i : \mathbb{B}_{C_3} \to \mathbb{B}_{S_3}$, $\widetilde{Ind}_{C_3}^{S_3} : \widetilde{R_{\mathbb{C}}C_3} \to R_{\mathbb{C}}S_3/D'_{S_3}$ and θ' still commute. This follows from the fact that

$$Ind_{C_3}^{S_3}(\varphi(\phi_{[a,a,a]_{C_3}})) \equiv Ind_{C_3}^{S_3}\rho_1 = \chi_2 \equiv \varphi(\phi_{[\hat{a}]_{S_3}}) \equiv \varphi(\phi_{\mathbb{B}_i([a,a,a]_{C_3})})$$

where the equivalence is taken modulo D'_{S_3}. ρ_1 denotes a one dimensional faithful representation of C_3.

5. The Case $\mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z}$

In this section we want to have a closer look at the case $G = \mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z}$, p an odd prime. We know by example 4 that $\mathbb{B}_G$ is isomorphic to $\mathbb{Z}^{\frac{p^2-1}{2}}$ and thus by theorem 21 and proposition 23, $\theta : \mathbb{B}_G \to \mathbb{A}_G$ is injective. The main purpose of this section is to prove the following theorem.

Theorem 25. *The index Δ of $\theta(\mathbb{B}_G)$ in $\mathbb{A}_G$ is $(h_p^-)^{p+1}\cdot p^i$, where i is an integer in the range $1-p+k \le i \le k+(p-1)^2/2$ for some $k\in\mathbb{N}$; h_p^- denotes the first factor of the class number of $\mathbb{Q}(e^{2\pi i/p})$; i.e., $\Delta = \left[\mathbb{A}_G : \theta(\mathbb{B}_G)\right] = (h_p^-)^{p+1}\cdot p^i$.*

Before we can prove this theorem, we need some lemmata. Let G_j, $j=0,\dots,p$, be the distinct cyclic subgroups of order p in G. In the sequel $\mathbb{B}_i$ will denote a map $\mathbb{B}_{G_j}\to\mathbb{B}_G$ coming from an inclusion $i: G_j\to G$. We can consider the following commutative diagram.

$$\begin{array}{ccccc}
\bigoplus \mathbb{B}_{G_j} & \xrightarrow{\oplus\mathbb{B}_i} & \mathbb{B}_G & \xrightarrow{\oplus\mathbb{B}res_{G_j}^G} & \bigoplus \mathbb{B}_{G_j} \\
{\scriptstyle\oplus\theta}\downarrow & & {\scriptstyle\theta}\downarrow & & {\scriptstyle\oplus\theta}\downarrow \\
\bigoplus \mathbb{A}_{G_j} & \xrightarrow{\oplus\widetilde{Ind}_{G_j}^G} & \mathbb{A}_G & \xrightarrow{\oplus\widetilde{Res}_{G_j}^G} & \bigoplus \mathbb{A}_{G_j}
\end{array}$$

Notation: In the sequel if we want to talk about the index of, for example, $\oplus\mathbb{B}res_{G_j}^G\circ\oplus\mathbb{B}_i(\bigoplus\mathbb{B}_{G_j})$ in $\bigoplus\mathbb{B}_{G_j}$ we will omit the maps and just write the index of $\bigoplus\mathbb{B}_{G_j}$ in $\bigoplus\mathbb{B}_{G_j}$.
By theorem 14 and proposition 15 the map $\oplus\theta$ is injective and the index of $\bigoplus\mathbb{B}_{G_j}$ in $\bigoplus\mathbb{A}_{G_j}$ is $(h_p^-)^{p+1}$. An easy computation shows that the map $\oplus\mathbb{B}res_{G_j}^G\circ\oplus\mathbb{B}_i$ is just multiplication by p, thus $\oplus\mathbb{B}_i$ is injective and the index of $\bigoplus\mathbb{B}_{G_j}$ in $\bigoplus\mathbb{B}_{G_j}$ is $p^{(p^2-1)/2}$. (By example 2, $\mathbb{B}_{G_j}\cong\mathbb{Z}^{(p-1)/2}$.) On the other hand we have also that the map $\oplus\mathbb{B}_i\circ\oplus\mathbb{B}res_{G_j}^G$ is multiplication by p and thus $\oplus\mathbb{B}res_{G_j}^G$ is injective and the index of $\mathbb{B}_G$ in $\mathbb{B}_G$ is also $p^{(p^2-1)/2}$.

Lemma 26. *The index of $\bigoplus\mathbb{B}_{G_j}$ in $\mathbb{B}_G$ is p^{p-1}.*

Proof. Recall the notations of examples 2 and 4 and let $G_j=\langle xy^j\rangle$, $j=0,\dots,p-1$ and $G_p=\langle y\rangle$. A basis for $\mathbb{B}_{G_j}$ is given by:

$$[x,x^k,x^{p-k-1}]_{G_0}\ ,\ k=1,\dots,(p-1)/2$$
$$[y,y^l,y^{p-l-1}]_{G_p}\ ,\ l=1,\dots,(p-1)/2$$
$$\left[xy^j,(xy^j)^r,(xy^j)^{p-r-1}\right]_{G_j}\ ,\ r=1,\dots,(p-1)/2\ ,\ j=1,\dots,p-1.$$

For $\mathbb{B}_G$ we have the following basis:

$$[x,x^k,x^{p-k-1}]_G=\mathbb{B}_i([x,x^k,x^{p-k-1}]_{G_0})\ ,\ k=1,\dots,(p-1)/2$$
$$[y,y^l,y^{p-l-1}]_G=\mathbb{B}_i([y,y^l,y^{p-l-1}]_{G_p})\ ,\ l=1,\dots,(p-1)/2$$
$$[x^j,y^i,x^{p-j}y^{p-i}]_G\ ,\ j=1,\dots,(p-1)/2\ ,\ i=1,\dots,p-1.$$

Thus we see that the basis elements of $\mathbb{B}_G$, $\left[x, x^k, x^{p-k-1}\right]_G$, $k = 1, \dots, (p-1)/2$, and $\left[y, y^l, y^{p-l-1}\right]_G$, $l = 1, \dots, (p-1)/2$, are induced form $\mathbb{B}_{G_0}$ and $\mathbb{B}_{G_p}$. Consequently they are zero in the quotient $Q = \mathbb{B}_G / \bigoplus \mathbb{B}_{G_j}$. It remains to find the relations between the other basis elements of $\mathbb{B}_G$ in Q.

We can write the images of the basis elements of $\mathbb{B}_{G_j}$, $j = 1, \dots, p-1$, as a linear combination in the basis elements of $\mathbb{B}_G$.

$$\begin{aligned}\mathbb{B}_i(\left[xy^j, (xy^j)^r, (xy^j)^{p-r-1}\right]_{G_j}) &= \ominus\left[x, y^j, (xy^j)^{p-1}\right] \ominus \left[x^r, y^{rj}, (xy^j)^{p-r}\right] \\ &\oplus\left[x^{r+1}, y^{j(r+1)}, (xy^j)^{p-r-1}\right] \oplus \left[x, x^r, x^{p-r-1}\right]_G \oplus \left[y^j, y^{jr}, y^{p-jr-j}\right]_G\end{aligned}$$

(Note that $\left[y^j, y^{jr}, y^{p-jr-j}\right]_G$ can be written as a linear combination in the $\left[y, y^l, y^{p-l-1}\right]_G$, $l = 1, \dots, (p-1)/2$.) If we consider the above equation in the quotient Q, we obtain the remaining relations between the basis elements of $\mathbb{B}_G$ in Q:

$$\left[x, y^j, (xy^j)^{p-1}\right]_G \oplus \left[x^r, y^{rj}, (xy^j)^{p-r}\right]_G \equiv \left[x^{r+1}, y^{j(r+1)}, (xy^j)^{p-r-1}\right]_G$$

$r = 1, \dots, (p-1)/2$, $j = 1, \dots, p-1$.
From this we conclude that the quotient Q is generated by the elements $\left[x, y^j, (xy^j)^{p-1}\right]_G$, $j = 1, \dots, p-1$ and thus the index is p^{p-1}. $\square$

From this lemma we deduce immediately that the index of $\mathbb{B}_G$ in $\bigoplus \mathbb{B}_{G_j}$ is $p^{(p-1)^2/2}$.

Lemma 27. *The maps* $\oplus\widetilde{Res}^G_{G_j} \circ \oplus\widetilde{Ind}^G_{G_j} : \bigoplus \mathbb{A}_{G_j} \to \bigoplus \mathbb{A}_{G_j}$ *and* $\oplus\widetilde{Ind}^G_{G_j} \circ \oplus\widetilde{Res}^G_{G_j} : \mathbb{A}_G \to \mathbb{A}_G$ *are just multiplication by* p.

Proof. Take a class $a_j \in \mathbb{A}_{G_j}$ with representative $\xi_j \in R_{\mathbb{C}}G_j$. The induced representation $Ind^G_{G_j}(\xi_j)$ consists of p copies of ξ_j and G/G_j operates by permutation on the p copies. Thus $Res^G_{G_l} \circ Ind^G_{G_j}(\xi_j)$ is $p \cdot a_j$ in $\mathbb{A}_{G_l}$ whenever $l = j$ and zero when $l \neq j$. This proves the assertion for the first map.

For the second map take a class $a \in \mathbb{A}_G$ with representative $\xi \in R_{\mathbb{C}}G$. Then restrict $\oplus Ind^G_{G_j} \circ \oplus Res^G_{G_j}(\xi)$ to a subgroup G_l of G and we obtain:

$$\begin{aligned}Res^G_{G_l} \circ \oplus Ind^G_{G_j} \circ \oplus Res^G_{G_j}(\xi) &= Res^G_{G_l} \circ \oplus_j (Ind^G_{G_j} \circ Res^G_{G_j})(\xi) = \\ \oplus_j Res^G_{G_l} \circ Ind^G_{G_j} \circ Res^G_{G_j}(\xi) &= p \cdot Res^G_{G_l}(\xi) \oplus \text{multiples of } \omega_{G_l}.\end{aligned}$$

The map $\oplus_l Res^G_{G_l}$ is injective on $R_{\mathbb{C}}G$ thus the map $\oplus Ind^G_{G_j} \circ \oplus Res^G_{G_j}$ acts by multiplication by p, up to multiples of ω_G on $R_{\mathbb{C}}G$ and consequently the map $\oplus\widetilde{Ind}^G_{G_j} \circ \oplus\widetilde{Res}^G_{G_j}$ acts by multiplication by p on $\mathbb{A}_G$. $\square$

An immediate consequence of this lemma is that the map $\oplus\widetilde{Ind}_{G_j}^{G}$ is injective and that the index of $\bigoplus \mathbb{A}_{G_j}$ in $\bigoplus \mathbb{A}_{G_j}$ is $p^{(p^2-1)/2}$. (Note that $\mathbb{A}_{G_j} \cong \mathbb{Z}^{(p-1)/2}$.) Furthermore the kernel of the map $\oplus\widetilde{Res}_{G_j}^{G} : \mathbb{A}_G \to \bigoplus \mathbb{A}_{G_j}$ is isomorphic to $(\mathbb{Z}/p\mathbb{Z})^k$ for some $k \in \mathbb{N}$.

Lemma 28. *The group $\mathbb{A}_G$ is isomorphic to $\mathbb{Z}^{\frac{p^2-1}{2}} \oplus (\mathbb{Z}/p\mathbb{Z})^k$ for some $k \in \mathbb{N}$.*

Proof. We know that $\bigoplus \mathbb{A}_{G_j}$ is isomorphic to $\mathbb{Z}^{(p^2-1)/2}$ and that it injects into $\mathbb{A}_G$. This proves that $\mathbb{Z}^{(p^2-1)/2}$ is a direct summand of $\mathbb{A}_G$. On the other hand the kernel of $\oplus\widetilde{Res}_{G_j}^{G} : \mathbb{A}_G \to \bigoplus \mathbb{A}_{G_j} \cong \mathbb{Z}^{(p^2-1)/2}$ is isomorphic to $(\mathbb{Z}/p\mathbb{Z})^k$ for some $k \in \mathbb{N}$ and thus the assertion holds. □

Proof of theorem 25. By example 4 the group $\mathbb{B}_G$ is isomorphic to $\mathbb{Z}^{\frac{p^2-1}{2}}$ and thus by theorem 21 the map $\theta : \mathbb{B}_G \to \mathbb{A}_G$ is injective. Lemma 28 tells us now that the index Δ is finite.

There are two possibilities to compute the index $\left[\mathbb{A}_G : \bigoplus \mathbb{B}_{G_j}\right]$. For the first possibility we use lemma 26.

$$\left[\mathbb{A}_G : \bigoplus \mathbb{B}_{G_j}\right] = \left[\mathbb{A}_G : \mathbb{B}_G\right] \cdot \left[\mathbb{B}_G : \bigoplus \mathbb{B}_{G_j}\right] = \Delta \cdot p^{p-1}$$

From lemma 28 we can also deduce that the index of $\bigoplus \mathbb{A}_{G_j}$ in $\mathbb{A}_G$ is some prime power p^i with $i = 0, \dots, k + (p^2-1)/2$, thus for the second possibility we get

$$\begin{aligned} \left[\mathbb{A}_G : \bigoplus \mathbb{B}_{G_j}\right] &= \left[\mathbb{A}_G : \bigoplus \mathbb{A}_{G_j}\right] \cdot \left[\bigoplus \mathbb{A}_{G_j} : \bigoplus \mathbb{B}_{G_j}\right] \\ &= p^i \cdot (h_p^-)^{p+1} \quad , \quad i = 0, \dots, k + (p^2-1)/2. \end{aligned}$$

Collecting the results of the two equations we obtain $\Delta = (h_p^-)^{p+1} \cdot p^i$ where i ranges over the integers $1 - p + k \le i \le k + (p-1)^2/2$. □

Remark 6. In the case when the prime p is 3 the index $\left[\mathbb{A}_G : \bigoplus \mathbb{A}_{G_j}\right]$ is also $p^{p-1} = 9$ thus $\Delta = (h_3^-)^4 = 1$.

In view of remark 6 we can formulate a conjecture.

$$\Delta = (h_p^-)^{p+1} \qquad \text{(Conjecture)}$$

To prove this conjecture it would be enough to show that $\left[\mathbb{A}_G : \bigoplus \mathbb{A}_{G_j}\right] = p^{p-1}$. In general this index is hard to compute, however for the prime 3 we have seen that the conjecture is true.

For regular primes p, i.e., p doesn't divide its class number h_p^-, we can give an equivalent statement to the conjecture.

Proposition 29. *Let p be a regular prime. Statements 1. and 2. are equivalent.*

1. (a) *Let a be an element of $\mathbb{A}_G$ such that its restriction to every subgroup G_j lies in the image of θ, i.e., $\widetilde{Res}^G_{G_j} a \in \theta(\mathbb{B}_{G_j})$, $j = 0, \dots, p$. Then the element a itself lies already in $\theta(\mathbb{B}_G)$.*
   (b) *Let a_j, $j = 0, \dots, p$, be elements in $\mathbb{A}_{G_j}$, such that the sum of their induced images lies in $\theta(\mathbb{B}_G)$, i.e., $\oplus \widetilde{Ind}^G_{G_j} a_j \in \theta(\mathbb{B}_G)$. Then the representations a_j lie already in $\theta(\mathbb{B}_{G_j})$.*
2. *The conjecture $\Delta = (h_p^-)^{p+1}$ is true.*

Proof. First we will prove 1.⇒2. .

Statement 1.(a) is equivalent to the fact that the induced map $\mathbb{A}_G/\mathbb{B}_G \to \bigoplus \mathbb{A}_{G_j}/\bigoplus \mathbb{B}_{G_j}$ is injective. Statement 1.(b) is equivalent to the fact that the induced map $\bigoplus \mathbb{A}_{G_j}/\bigoplus \mathbb{B}_{G_j} \to \mathbb{A}_G/\mathbb{B}_G$ is injective. From this we conclude $\Delta|(h_p^-)^{p+1}$ and $(h_p^-)^{p+1}|\Delta$ which in turn proves the conjecture.

Next we will prove 2.⇒1. .

The map $\oplus \widetilde{Ind}^G_{G_j} \circ \oplus \widetilde{Res}^G_{G_j}$ is multiplication by p. Thus the induced map $\mathbb{A}_G/\mathbb{B}_G \to \mathbb{A}_G/\mathbb{B}_G$ is injective as p doesn't divide the class number h_p^-. From this we conclude that the map $\mathbb{A}_G/\mathbb{B}_G \to \bigoplus \mathbb{A}_{G_j}/\bigoplus \mathbb{B}_{G_j}$, which is induced by $\oplus \widetilde{Res}^G_{G_j}$, is injective, which is equivalent to 1.(a).

The same argument using the map $\oplus \widetilde{Res}^G_{G_j} \circ \oplus \widetilde{Ind}^G_{G_j}$ shows that 1.(b) is also true. □

Remark 7. Statement 1. of proposition 29 is never true if the prime p is irregular.

6. G-Equivariant Cobordism

In this section we will define a map χ from the G-equivariant cobordism group of surface diffeomorphisms Ω_G to $\mathbb{B}_G$ and prove that this map is surjective and describe its kernel. We will only talk about oriented cobordism, thus every diffeomorphism is orientation preserving. First we need some definitions.

Definition 30. Let M_1 (resp. M_2) be a compact, oriented, connected Riemann surface with smooth G-action $\kappa_1 : G \to \mathit{Diffeo}_+(M_1)$ (resp. $\kappa_2 : G \to \mathit{Diffeo}_+(M_2)$) We say that κ_1 is G-equivariant cobordant to κ_2, written $\kappa_1 \sim \kappa_2$, if there exists a smooth, compact, oriented, connected 3-manifold V and a smooth G-action Φ on V such that

1. The boundary of V is the disjoint union of M_1 and $-M_2$, $\partial(V) = M_1 \cup -M_2$. The notation $-M_2$ denotes M_2 with opposite orientation. The orientations on M_1 and $-M_2$ coincide with the one induced by V.
2. Φ restricted to $\partial(V)$ agrees with $\kappa_1 \cup \kappa_2$.

We also say that κ_1 is zero G-equivariant cobordant, written $\kappa_1 \sim 0$, if $\partial(V) = M_1$. Ω_G will denote the set of G-equivariant cobordism classes and a class will be denoted by (κ, M).

There is an addition in Ω_G given by the G-equivariant connected sum defined in section 2. This addition is well defined as it doesn't depend on the representative of the G-equivariant cobordism classes, thus we have an Abelian monoid. In fact we have even more, Ω_G is an Abelian group. Indeed, for any element (κ, M), there exists also its inverse element $(\kappa, -M)$ in Ω_G. The zero element is the zero G-equivariant cobordism class.

Lemma 31. *Let κ_1 and κ_2 be G-equivariant cobordant. Then they have the same singular orbit data.*

Proof. There is a compact, connected smooth 3-manifold V together with a smooth G-action Φ on V such that $\partial(V) = M_1 \cup -M_2$ and Φ restricted to $\partial(V)$ agrees with $\kappa_1 \cup \kappa_2$.
Let $x \in M_1$ and Gx be a singular orbit such that $\gamma \in G$ generates the stabilizer of x and acts by rotation through $2\pi/|\langle\gamma\rangle|$. The action of $H = \langle\gamma\rangle$ extends to V by extending the fixed point x to a properly embedded arc of fixed points of H. The endpoint y of this arc has to lie in a boundary component, thus either in M_1 or in M_2. If y lies in M_1 we have again two possibilities.

Firstly y lies in the same singular orbit Gx, but then there exists $g \in G$ with $gx = y$ and thus g maps the arc onto itself and the diffeomorphism given by g isn't orientation preserving anymore. Secondly y lies in another singular orbit Gy and thus γ acts by rotation through $-2\pi/|\langle\gamma\rangle|$ on y. The arc from x to y is mapped by g to arcs from gx to gy. Thus Gx and Gy are two singular orbits on M_1 which cancel.

If y lies in M_2 then by the same arguments as above and the fact that we have the opposite orientation on M_2 we deduce that Gx and Gy are the same singular orbits.

In this way we can show that up to cancelling pairs κ_1 and κ_2 have the same singular orbits. □

By lemma 31 we can now define a homomorphism χ which sends a G-equivariant cobordism class to its singular orbit data.

$$\chi : \Omega_G \to \mathbb{B}_G.$$

Theorem 32. *The map χ is surjective and the kernel consists of cobordism classes of free G-actions.*

Proof. First we will prove the statement about the kernel. Let κ denote a G-action on a surface M whose image in $\mathbb{B}_G$ is zero, i.e., the singular orbits cancel. We will show that κ is cobordant to a free G-action.

First we take the product cobordism $V = M \times [0,1]$, where G is extended over V in the obvious way. Next we modify V on the top end $M \times \{1\}$ in a similar way as we did to introduce relations in W_G in section 2. Suppose Gx and Gy are cancelling singular orbits on $M \times \{1\}$ with stabilizer $\langle \gamma \rangle$ at x and y. Let T be a set of representatives for the $\langle \gamma \rangle$ left cosets of G. Find discs D_1 and D_2 around x and y respectively such that D_j is fixed by $\langle \gamma \rangle$, $j = 1, 2$, and $\cup_{j=1,2} \cup_{t \in T} \{tD_j\}$ are mutually disjoint. Then connect each pair of discs tD_1, tD_2 by a solid tube $D^2 \times [0,1]$ for every element $t \in T$. Let V_1 denote the 3-manifold which results after connecting all the cancelling pairs of singular orbits by solid tubes. The action of G can be extended to V_1 by rotating and permuting the solid tubes. The manifold V_1 provides the G-equivariant cobordism between κ and a free G-action.

To prove that χ is surjective just note that every singular orbit data gives rise to a G-action with the appropriate singular orbits and thus to the corresponding G-equivariant cobordism class. $\square$

With the surjection χ the map θ becomes a G-signature. It has common properties with the G-signature of Atiyah and Singer; for example, it is injective on the copies of $\mathbb{Z}$ in $\mathbb{B}_G$. There is one essential difference, however: the G-signature given by θ can also detect real representations (see also remark 5).

Proposition 33. *The map χ is an isomorphism whenever $G \cong \mathbb{Z}/n\mathbb{Z}$, n a positive integer.*

Proof. Conner and Floyd prove in their book [4] (see also [5, proposition 3.1]) that the cobordism group of free $\mathbb{Z}/n\mathbb{Z}$-actions is isomorphic to $H_2(\mathbb{Z}/n\mathbb{Z}; \mathbb{Z})$ which is zero for every n. Thus every free G-action is zero G-equivariant cobordant and by theorem 32 χ is an isomorphism. $\square$

As was already mentioned earlier, in [10] the author gives a complete computation of the group $\mathbb{B}_G$. This means we can compute the G-equivariant cobordism group in dimension two, up to the classes of free actions, for any finite group G.

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